BV-STRUCTURES ON THE HOMOLOGY OF THE FRAMED LONG KNOT SPACE

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Abstract. We introduce BV-algebra structures on the homology of the space of framed long knots in $\mathbb{R}^n$ in two ways. The first one is given in a similar fashion to Chas-Sullivan's string topology [5]. The second one is defined on the Hochschild homology associated with a cyclic, multiplicative operad over graded modules. The latter can be applied to a certain spectral sequence converging to the homology of the space of framed long knots. Conjecturally these two structures coincide with each other.

1. Introduction

The space of framed long embeddings is known to be acted on by the little disks operad [2]. A natural question is whether this action extends to any action of the *framed* little disks operad. The answer seems affirmative, in view of [3, 12, 13], [6, 14]. In fact, Paolo Salvatore told the author that he realized in his draft an action of the framed little disks operad on the space of framed long knots, using his solution to the topological cyclic Deligne conjecture [13].

The first result of this paper (Theorem 3.5) is a geometric and homological counterpart to Salvatore's homotopy-theoretical action. We imitate Chas-Sullivan's string topology [5] to define a *BV-algebra* structure on the homology of the space of framed long knots. Our BV-structure is outlined as follows. The bracket (which we call *Poisson bracket*) is induced by an action of little disks operad [2] mentioned above. The *BV-operation* (usually denoted by Δ) is derived from Hatcher's cycle [7, p.3], which in a sense "pushes the base point along the long knots." As a corollary we obtain a Lie algebra structure on the S^1-equivariant homology.

The second result is an algebraic one. Based on [13, 1], we can construct a homology spectral sequence converging to the homology of the space of framed long knots (at least in the higher-codimension cases). Its E^2-term is the *Hochschild homology* associated with the homology operad $H_*(f\mathcal{C})$ of framed little disks. $H_*(f\mathcal{C})$ is cyclic [3] and multiplicative. In §4, we prove that a graded version of Connes' boundary operator (see [8]) induces a BV-structure on such a Hochschild homology. Our proof is a direct analogue to that for non-graded case (see [10] for details) and can be applied to any cyclic operad other than the framed disks. The formula of our BV-operation would be obtained by Salvatore's action mentioned above.

2. Notations

We denote the standard basis of the vector space $\mathbb{R}^n$ by

$$e_1 := (1, 0, \dots, 0), \quad e_2 := (0, 1, 0, \dots, 0), \ \dots, \ e_n := (0, \dots, 0, 1).$$

Date: October 12, 2011.

2010 *Mathematics Subject Classification.* Primary 55P50, Secondary 16E40, 57Q45, 57R40.

$B^n := \{x \in \mathbb{R}^n \,|\, |x| \le 1\}$ denotes the n-ball, and $S^n := \partial B^{n+1} \subset \mathbb{R}^{n+1}$. We often write $\infty := e_{n+1} \in S^n$. We always regard S^1 as $\mathbb{R}/\mathbb{Z}$. We fix the Gram-Schmidt orthonormalization and denote it by $GS : GL_N(\mathbb{R}) \xrightarrow{\simeq} SO(N)$.

Definition 2.1. A *based knot* in S^n is an embedding $\varphi : S^1 \hookrightarrow S^n$ such that

$$\varphi(0) = \infty, \quad \varphi'(0)/|\varphi'(0)| = e_1.$$

A *framed based knot* in S^n is a based knot φ together with a smooth map $w : S^1 \to SO(n+1)$ such that

- if we write $w(t) = (w_0(t), \dots, w_n(t))$, $w_i(t) \in S^n$, then for any $t \in S^1$,
$$w_0(t) = \varphi'(t)/|\varphi'(t)|, \quad w_n(t) = \varphi(t),$$
- $w(0) = I_{n+1}$ (the identity matrix).

Define

$$\widetilde{\mathrm{Emb}}_{**}(S^1, S^n) := \{(\varphi; w) \,|\, \text{framed, based knot in } S^n\}.$$

We call such a (φ, w) as above a framed knot, since the first n columns of $w(t)$ gives an orthonormal frame of $T_{\varphi(t)}S^n$.

Definition 2.2 ([2])**.** For a manifold M, define the space $\mathrm{EC}(k, M)$ by

$$\mathrm{EC}(k, M) := \{f : \mathbb{R}^k \times M \hookrightarrow \mathbb{R}^k \times M \,|\, f|_{\{|t| \ge 1\} \times M} = \mathrm{id}\}.$$

We can think of $f \in \mathrm{EC}(1, B^{n-1})$ as a framed long knot in $\mathbb{R}^n$.

Proposition 2.3. $\mathrm{EC}(1, B^{n-1})$ *is homotopy equivalent to* $\widetilde{\mathrm{Emb}}_{**}(S^1, S^n)$.

Proof. The above homotopy equivalence is given by $\mathrm{EC}(1, B^{n-1}) \ni f \mapsto (\varphi; w) \in \widetilde{\mathrm{Emb}}_{**}(S^1, S^n)$, where

$$\varphi(t) := p \circ f \circ l(t), \quad w(t) := GS\Big(\varphi'(t), \Big(p_* \frac{\partial f}{\partial x_i}(l(t), 0)\Big)_{i=1}^{n-1}, \varphi(t)\Big)$$

for suitable diffeomorphisms $l : (0, 1) \to (-2, 2)$ and $p : (-2, 2) \times \mathrm{Int}\, B^{n-1} \to S^n \setminus \infty$. We can choose p so that φ is smooth at $t = 0$. $\square$

Therefore if some algebraic structure is imposed on the homology of one of the above spaces, then the homology of the other also admits the same one.

3. A geometric BV-structure

Definition 3.1. A *k-Poisson algebra A* is a graded commutative algebra equipped with a graded Lie bracket $[-, -] : A \times A \to A$ of degree k (called a *Poisson bracket*) satisfying

$$[x, yz] = [x, y]z + (-1)^{(\tilde{x}+k)\tilde{y}} y[x, z],$$

where $\tilde{x}$ is the degree of x, that is, $x \in A_{\tilde{x}}$. A 1-Poisson algebra A is called a *BV-algebra* if it is equipped with an operation $\Delta : A \to A$ of degree one satisfying

- $\Delta \circ \Delta = 0$,
- $\Delta(xy) = \Delta(x)y + (-1)^{\tilde{x}} x\Delta(y) + (-1)^{\tilde{x}}[x, y]$.

The aim of this section is to give $H_*(\widetilde{\mathrm{Emb}}_{**}(S^1, S^n))$ a BV-algebra structure. The Poisson algebra structure on $H_*(\mathrm{EC}(1, B^{n-1}))$ has already defined in [3]. First we re-interpret this on $H_*(\widetilde{\mathrm{Emb}}_{**}(S^1, S^n))$ (§3.1.1, 3.1.2). Then we define the Δ-operation using an S^1-action on $\widetilde{\mathrm{Emb}}_{**}(S^1, S^n)$.

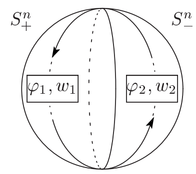

FIGURE 3.1. Connected-sum on $\widetilde{\mathrm{Emb}}_{**}(S^1, S^n)$

3.1. **Poisson structure.** The *connected-sum* operation (denoted by $\sharp$) makes $\mathrm{EC}(k, M)$ an H-space. For any $f, g \in \mathrm{EC}(k, M)$, $f\sharp g$ is defined by "stacking" g after f. In fact $\sharp$ is homotopy commutative, since $\sharp$ is a part of an action of the little $(k+1)$-disks operad on the space $\mathrm{EC}(k, M)$ [2]. The main idea of such an action can be found in [2, Figure 2]. As a corollary we have the following.

Theorem 3.2 ([2])**.** *$H_*(\mathrm{EC}(k, M))$ admits a k-Poisson algebra structure.*

Below we will consider the product $x \cdot y$ (or simply xy) and the bracket $\lambda(x, y)$ defined on $H_*(\mathrm{EC}(1, B^{n-1}))$ and describe them on $H_*(\widetilde{\mathrm{Emb}}_{**}(S^1, S^n))$.

3.1.1. *Connected-sum on* $\widetilde{\mathrm{Emb}}_{**}(S^1, S^n)$. Figure 3.1 defines the connected-sum on $\widetilde{\mathrm{Emb}}_{**}(S^1, S^n)$ up to homotopy. A little more precisely, let $S^n_\pm := \{(x_1, \dots, x_{n+1}) \in S^n \mid \pm x_1 > 0\}$ and let $\Psi_\pm : S^n \setminus \{\infty\} \xrightarrow{\cong} S_\pm$ be diffeomorphisms defined by

$$\Psi_\pm(x_1, \dots, x_{n+1}) := R_\pm\left(kx_1, \dots, kx_n, (x_{n+1} - 1)/2\right),$$

where $k = \sqrt{(3 - x_{n+1})/4(1 - x_{n+1})}$ and $R_\pm$ are the $\pm\pi/2$-rotations in $x_1 x_{n+1}$-plane. For any $\sigma_i = (\varphi_i; w_i) \in \widetilde{\mathrm{Emb}}_{**}(S^1, S^n)$ $(i = 1, 2)$, define $\Theta'(\sigma_1, \sigma_2) : S^1 \to S^n \times SO(n+1)$ by

$$\Theta'(\sigma_1, \sigma_2)(t) := \begin{cases} (\Psi_+\varphi_1(2t); GS\Psi_{+*}w_1(2t)) & 0 \le t \le 1/2, \\ (\Psi_-\varphi_2(2t-1); GS\Psi_{-*}w_2(2t-1)) & 1/2 \le t \le 1. \end{cases}$$

One can modify Θ' so that it gives a smooth map at $t = 0, 1/2$. Then the modified map Θ induces the product on $H_*(\widetilde{\mathrm{Emb}}_{**}(S^1, S^n))$, which coincides with that on $H_*(\mathrm{EC}(1, B^{n-1}))$ induced by the connected sum.

3.1.2. *Poisson bracket for* $\widetilde{\mathrm{Emb}}_{**}(S^1, S^n)$. Poisson bracket λ can be written by using the *star-operation*, denoted by $*$, defined on chains. This is given by the right-half of [2, Figure 2]. Namely, $f * g$ begins with the connected-sum $f\sharp g$, then "push off" g through f as in the right-half of [2, Figure 2], ending at $g\sharp f$.

More precisely, $*$ for $\widetilde{\mathrm{Emb}}_{**}(S^1, S^n)$ is induced by the map

$$* : I \times \widetilde{\mathrm{Emb}}_{**}(S^1, S^n)^{\times 2} \to \widetilde{\mathrm{Emb}}_{**}(S^1, S^n)$$

defined as follows. Let $\sigma = (\varphi_\sigma; w_\sigma)$ $(w_\sigma = (w_{\sigma,0}, \dots, w_{\sigma,n}))$ be a framed based knot. There is a positive number $\epsilon = \epsilon_\sigma > 0$ (depending continuously on σ) such that the map $\check{\varphi}_\sigma : S^1 \times B^{n-1} \to S^n$ defined by

$$\check{\varphi}_\sigma(t; x) := \varphi_\sigma(t) + sp\big(\epsilon \sum_{1 \le i \le n-1} x_i w_{\sigma,i}(t)\big) \quad (x = (x_1, \dots, x_{n-1}) \in B^{n-1})$$

is an embedding, where sp is the stereographic projection $T_{\varphi_\sigma(t)}S^n \to S^n \setminus \{-\varphi_\sigma(t)\}$.

For any $\alpha \in S^1$, choose a monotonously increasing C^∞-function $\eta = \eta_{\alpha,\epsilon} : [-2,2] \xrightarrow{\cong} [\alpha-\epsilon, \alpha+\epsilon]$ such that

$$\eta(t) = t + \alpha \pm (\epsilon - 2) \quad \text{for} \quad |t \pm 2| < \epsilon$$

and define $\overline{\varphi}_{(\sigma,\alpha)} : [-2,2] \times B^{n-1} \to S^n$ by

$$\overline{\varphi}_{(\sigma,\alpha)}(t;x) := \check{\varphi}_\sigma(\eta(t),x).$$

Let $\tau \in \widetilde{\mathrm{Emb}}_{**}(S^1,S^n)$ be another framed based knot and $\hat{\tau} \in \mathrm{EC}(1,B^{n-1})$ the corresponding framed long knot under the homotopy equivalence from Proposition 2.3. Then define $\sigma *_\alpha \tau = (\tilde{\varphi};\tilde{w}) \in \widetilde{\mathrm{Emb}}_{**}(S^1,S^n)$ by

$$(\tilde{\varphi}(t);\tilde{w}(t)) :=$$
$$\begin{cases} (\varphi_\sigma(t); w_\sigma(t)) & |t-\alpha| > \epsilon, \\ \Big(\overline{\varphi}_{(\sigma,\alpha)}(\hat{\tau}(\eta^{-1}(t),0)); GS\Big(\dfrac{\partial(\overline{\varphi}_{(\sigma,\alpha)}(\hat{\tau}(\eta^{-1}(t),0)))}{\partial x_i}\Big)_{i=0}^n\Big) & |t-\alpha| \le \epsilon. \end{cases}$$

For p- and q-cube chains x,y of $\widetilde{\mathrm{Emb}}_{**}(S^1,S^n)$, define a cube-chain $x*y$ by

$$x*y : I \times I^p \times I^q \to \widetilde{\mathrm{Emb}}_{**}(S^1,S^n), \quad (\alpha,\xi,\eta) \mapsto x(\xi) *_{-\alpha'} y(\eta),$$

where we put $\alpha' := (1-2\epsilon)\alpha + \epsilon$ so that $x*y$ starts with $x\sharp y$ and ends at $y \sharp x$.

Lemma 3.3. *On chains, we have $\lambda(x,y) \sim (-1)^{\tilde{x}-1}(x*y + (-1)^{\tilde{x}\tilde{y}} y*x)$.*

3.2. **BV-operation.** We define an S^1-action on $\widetilde{\mathrm{Emb}}_{**}(S^1,S^n)$ as was done in [7, p.3]. This action induces our BV-operation on $H_*(\widetilde{\mathrm{Emb}}_{**}(S^1,S^n))$.

For any $(\varphi;w) \in \widetilde{\mathrm{Emb}}_{**}(S^1,S^n)$ and $\alpha \in S^1$, define $(\varphi;w)^\alpha \in \widetilde{\mathrm{Emb}}_{**}(S^1,S^n)$ by

$$(\varphi;w)^\alpha(t) := (A\varphi(t-\alpha); Aw(t-\alpha)),$$

where $A = A(\alpha,\varphi,w) \in SO(n+1)$ (acting on S^n in the usual way) is the unique matrix satisfying $Aw(-\alpha) = I_{n+1}$ (and hence $A\varphi(-\alpha) = \infty$).

Lemma 3.4. *The above formula defines an S^1-action on $\widetilde{\mathrm{Emb}}_{**}(S^1,S^n)$. That is, $((\varphi;w)^\alpha)^\beta = (\varphi;w)^{\alpha+\beta}$, $(\varphi;w)^0 = (\varphi;w)$.*

This action induces our Δ-operation

$$\Delta : H_*(\widetilde{\mathrm{Emb}}_{**}(S^1,S^n)) \to H_{*+1}(\widetilde{\mathrm{Emb}}_{**}(S^1,S^n)).$$

We have $\Delta^2 = 0$ since Δ is induced by an S^1-action and $H_*(S^1) = \bigwedge(\iota)$, $\deg \iota = 1$.

Theorem 3.5. *$(H_*(\widetilde{\mathrm{Emb}}_{**}(S^1,S^n)), \cdot, \lambda, \Delta)$ is a BV-algebra.*

Proof. What we need to prove is that Δ is a derivation with respect to the product modulo λ;

$$\Delta(xy) - \Delta(x)y - (-1)^{\tilde{x}} x\Delta(y) = (-1)^{\tilde{x}}\lambda(x,y). \tag{3.1}$$

This is proved in a similar way as [5, Lemma 5.2]. Define two operations Δ_i $(i=1,2)$ on chains as the "first/last half" of Δ;

$$\Delta_i(x) : I \times I^p \to \widetilde{\mathrm{Emb}}_{**}(S^1,S^n), \quad \Delta_i(x)(u,\xi) = x(\xi)^{(u+i-1)/2}.$$

Let $\Delta^2 := \{0 \le v \le u \le 1\}$ be the standard 2-simplex and x,y be p- and q-chains. Define $\Phi_{x,y} : \Delta^2 \times I^p \times I^q \to \widetilde{\mathrm{Emb}}_{**}(S^1,S^n)$ by

$$\Phi_{x,y}((u,v),\xi,\eta) := ((x *_{-u'} y)(\xi,\eta))^{v''},$$

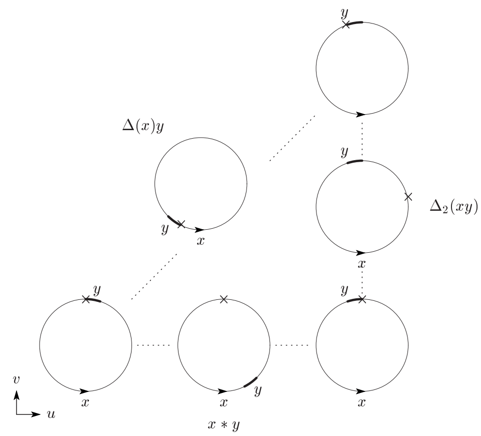

FIGURE 3.2. The chain $\Phi_{x,y}$; the symbol $\times$ is the base-point

where $v'' := (1-2\epsilon)v$. We observe that $\Phi_{x,y}$ is homologous to

- $x * y$ if restricted to $\{v = 0\}$,
- $\Delta_2(xy)$ if restricted to $\{u = 1\}$, and
- $\Delta(x)y$ if restricted to $\{u = v\}$ (see Figure 3.2).

Thus we have

$$x * y + \Delta_2(xy) - \Delta(x)y \sim \pm(\partial\Phi_{x,y} - \Phi_{\partial x,y} - (-1)^p \Phi_{x,\partial y}). \tag{3.2}$$

We also observe that

$$\Delta_2(yx) \sim (-1)^{pq}\Delta_1(xy), \quad \Delta_1(z) + \Delta_2(z) \sim \Delta(z) \quad \text{for any } x, y, z. \tag{3.3}$$

(3.2) and (3.3) imply (3.1). □

Proposition 3.6. *Δ is nontrivial when $n \geq 3$ is odd.*

Proof. It is an easy consequence of the third equation from Definition 3.1 that at least one of $\Delta(xy)$, $\Delta(x)$ and $\Delta(y)$ is not zero if $\lambda(x, y) \neq 0$. The nontriviality of λ is proved in [4] (when $n = 3$) and in [11] (when $n > 3$ is odd). □

Remark 3.7. The similar construction for $\mathrm{EC}(k, D^{n-k})$ is possible (an analogue to *brane topology* for embeddings).

3.3. **The string bracket.** Following [5, §6], consider the principal S^1-bundle

$$\pi : ES^1 \times \widetilde{\mathrm{Emb}}_{**}(S^1, S^n) \to ES^1 \times_{S^1} \widetilde{\mathrm{Emb}}_{**}(S^1, S^n).$$

Let $p : E \to ES^1 \times_{S^1} \widetilde{\mathrm{Emb}}_{**}(S^1, S^n)$ be the vector bundle of rank two associated with π, and E_0 the complement of the zero section of E. The Gysin exact sequence for p can be written as

$$\begin{aligned} \cdots \to H_i(\widetilde{\mathrm{Emb}}_{**}(S^1, S^n)) &\xrightarrow{\mathrm{E}} H_i^{S^1}(\widetilde{\mathrm{Emb}}_{**}(S^1, S^n)) \\ &\xrightarrow{c} H_{i-2}^{S^1}(\widetilde{\mathrm{Emb}}_{**}(S^1, S^n)) \xrightarrow{\mathrm{M}} H_{i-1}(\widetilde{\mathrm{Emb}}_{**}(S^1, S^n)) \to \dots \end{aligned}$$

(E is induced by $E_0 \hookrightarrow E$, c is given by using Thom isomorphism, and M is the connecting homomorphism). Define a bracket $\{-, -\}$ on $H_*^{S^1}(\widetilde{\mathrm{Emb}}_{**}(S^1, S^n))$ by

$$\{x, y\} := (-1)^{\tilde{x}}\mathrm{E}(\mathrm{M}(x)\mathrm{M}(y)).$$

The following is a corollary of Theorem 3.5 and proved in the same way as [5, Theorem 6.1].

Corollary 3.8. *$\{-,-\}$ is a graded Lie bracket of degree two.*

4. BV-structure on the Hochschild homology

This section can be read independently of the previous one. In this section we define a BV-algebra structure on *Hochschild homology* $HH_*(\mathcal{O})$ associated with a *cyclic multiplicative operad* $\mathcal{O}$ (in the category of *graded* modules).

One motivation is as follows. The space $\mathrm{EC}(1, B^{n-1})$ is weakly equivalent to the homotopy totalization of an operad which is weakly equivalent to the framed little n-disks operad $f\mathcal{C}_n$ [12]. There is a spectral sequence [1] converging to the homology of a homotopy totalization of a multiplicative operad over spaces ($f\mathcal{C}$ is equivalent to one of such operads). For $\mathrm{EC}(1, B^{n-1})$, the E^2-term of the spectral sequence is the Hochschild homology associated with the operad $H_*(f\mathcal{C})$. In general, for any multiplicative operad $\mathcal{O}$ over modules, $HH_*(\mathcal{O})$ admits a Poisson algebra structure [15], which is proved in Salvatore's draft to coincide with that described in [2]. Moreover if $\mathcal{O}$ is a cyclic operad (over *ungraded* modules), then $HH_*(\mathcal{O})$ admits a BV-algebra structure [16, 10]. In our case, $f\mathcal{C}$ is equivalent to a cyclic operad (of "conformal n-balls") [3], and it turns out that $H_*(f\mathcal{C})$ is a cyclic multiplicative operad (over *graded* modules). So it is natural to ask whether $HH_*(\mathcal{O})$ admits a suitable BV-algebra structure when $\mathcal{O}$ is a cyclic operad over *graded* modules. Note that our BV-structure would be derived from Salvatore's action on the homotopy totalization.

As for the operads, we follow the convention of [9].

4.1. **Hochschild homology.** For an operad $\mathcal{O}$ and $x \in \mathcal{O}(l)$, $y \in \mathcal{O}(m)$, define

$$x \circ_i y := x(\mathrm{id}, \dots, \mathrm{id}, y, \mathrm{id}, \dots, \mathrm{id}) \in \mathcal{O}(l+m-1),$$

where y sits in the i-th place, and $\mathrm{id} \in \mathcal{O}(1)$ is the identity element. When $\mathcal{O}$ is an operad of graded modules, we denote by $\tilde{x}$ the grading of x in the graded module $\mathcal{O}(l)$, that is, $x \in \mathcal{O}(l)_{\tilde{x}}$.

Let $\mathcal{O}$ be a multiplicative operad [9, Definition 10.1] over graded modules. We denote the multiplication by $\mu \in \mathcal{O}(2)$. The collection $\mathcal{O} = \{\mathcal{O}(k)\}_{k \geq 0}$ admits a structure of a cosimplicial module; the cosimplicial structure maps

$$d^i : \mathcal{O}(k-1) \to \mathcal{O}(k), \quad s^i : \mathcal{O}(k+1) \to \mathcal{O}(k)$$

($0 \leq i \leq k$) are defined as in [9, §10] by using μ and the unit element $e \in \mathcal{O}(0)$. The grading-preserving map

$$\partial_k : \mathcal{O}(k) \to \mathcal{O}(k+1), \quad \partial_k := d^0 - d^1 + \dots + (-1)^{k+1} d^{k+1}$$

satisfies $\partial_{k+1}\partial_k = 0$. We call the complex $\{\mathcal{O}, \partial\}$ the *Hochschild complex associated with $\mathcal{O}$*.

Define the *normalized Hochschild complex* $\tilde{\mathcal{O}}$ by

$$\tilde{\mathcal{O}}(k) := \bigcap_{0 \leq i \leq k-1} \ker\{s^i : \mathcal{O}(k) \to \mathcal{O}(k-1)\}.$$

The following is a well-known fact.

Lemma 4.1. *The map ∂_k restricts to $\partial_k : \tilde{\mathcal{O}}(k) \to \tilde{\mathcal{O}}(k+1)$ and the inclusion map $\tilde{\mathcal{O}} \to \mathcal{O}$ is a quasi-isomorphism.*

A Poisson algebra structure on the Hochschild homology $HH(\mathcal{O}) := H_*(\mathcal{O}, \partial)$ was defined in [15]; for $x \in \mathcal{O}(l)$ and $y \in \mathcal{O}(m)$, define the *degree* of x by

$$|x| := \tilde{x} - l$$

(this agrees with the homological degree in the spectral sequence) and operations

$$x \bullet y := (-1)^{l\tilde{y}} \mu(x, y),$$
$$[x, y] := x \bar{\circ} y - (-1)^{(|x|+1)(|y|+1)} y \bar{\circ} x,$$

where $\bar{\circ}$ is the map which should be compared with the star-operation $*$ (§3.1.2);

$$x \bar{\circ} y := \sum_{1 \le i \le l} (-1)^{(m-1)(l-i)+(l-1)b} x \circ_i y.$$

Theorem 4.2 ([15]). *For a multiplicative operad $\mathcal{O}$ over graded modules, $(HH(\mathcal{O}), \bullet, [\cdot, \cdot])$ is a Poisson algebra with respect to the degree $|\cdot|$.*

4.2. **Connes' boundary operation.** Suppose in addition that $\mathcal{O}$ is a *cyclic multiplicative operad* (see [10, Definition 3.11]); namely, there are grading-preserving linear maps

$$\tau_k : \mathcal{O}(k) \to \mathcal{O}(k)$$

satisfying $\tau_k^{k+1} = \mathrm{id}$, $\tau_0(e) = e$, $\tau_2(\mu) = \mu$ and, for $x \in \mathcal{O}(l)$ and $y \in \mathcal{O}(m)$,

$$\tau_{l+m}(x \circ_i y) = \begin{cases} \tau_l(x) \circ_{i-1} y & i \ge 2, \\ (-1)^{\tilde{x}\tilde{y}} \tau_m(y) \circ_m \tau_l(x) & i = 1. \end{cases}$$

Lemma 4.3 ([10, Theorem 1.4 (a)]). *Let $\mathcal{O}$ be a cyclic multiplicative operad over graded modules. The collection $\{\tau_k\}_{k \ge 0}$ of maps makes the cosimplicial module $\mathcal{O}$ into a cocyclic module; that is, for $1 \le i \le k$, we have*

$$\tau_k d^i = d^{i-1} \tau_{k-1}, \quad \tau_k s^i = s^{i+1} \tau_{k+1}.$$

Define the operation $B_k : \mathcal{O}(k) \to \mathcal{O}(k-1)$ by

$$B_k(x) := (-1)^{\tilde{x}} \sum_{1 \le i \le k} (-1)^{i(k-1)} \tau_{k-1}^{-i} s^{k-1} \tau_k (1 - \tau_k)(x).$$

This map is called *Connes' boundary operation* (for non-graded simplicial version, see [8, (2.1.7.1)]). Indeed B is a boundary map:

Lemma 4.4 ([8, §2]). *We have $B_k B_{k+1} = 0$, $B_{k+1} \partial_k = -\partial_{k-1} B_k$.*

Note that τ_k does not descend to a map on $\tilde{\mathcal{O}}(k)$. But the following holds.

Lemma 4.5 ([8, §2]). *B_k restricts to a map $B_k : \tilde{\mathcal{O}}(k) \to \tilde{\mathcal{O}}(k-1)$ of the form*

$$B_k(x) = (-1)^{\tilde{x}} \sum_{1 \le i \le k} (-1)^{i(k-1)} \tau_{k-1}^{-i} \sigma_k(x),$$

where $\sigma_k := s^{k-1} \tau_k$.

The latter statement follows from $s^{k-1} \tau_k^2 = \tau_{k-1} s^0$, which is a consequence of Lemma 4.3.

We have the induced map B_k on Hochschild homology by Lemma 4.4. The main result of this section is the following.

Theorem 4.6. *$(HH(\mathcal{O}), \bullet, [\cdot, \cdot], B)$ is a BV-algebra with respect to the grading $|\cdot|$.*

This theorem is known for cyclic multiplicative operads over ungraded modules [16], [10, §6]. The proof below is exactly same as that in [10, §6] when the degrees a and b are both even.

Proof. Let $x \in \tilde{\mathcal{O}}(l)_a$, $y \in \tilde{\mathcal{O}}(m)_b$. Define $Z(x,y) \in \tilde{\mathcal{O}}(l+m-1)_{a+b}$ by

$$Z(x,y) := (-1)^{|x||y|+a+b} \sum_{1\le j\le l} (-1)^{j(l+m-1)} \tau_{l+m-1}^{-j} \sigma_{l+m}(y \bullet x)$$

and define $H(x,y) \in \tilde{\mathcal{O}}(l+m-2)_{a+b}$ by $H(x,y) := \sum_{1\le j\le p\le l-1} H_{j,p}(x,y)$, where

$$H_{j,p}(x,y) := (-1)^{j(l-1)+(m-1)(p+1+l)+lb} \tau_{l+m-2}^{-j} \sigma_{l+m-1}(x \circ_{p-j+1} y).$$

It is not difficult to see that the following three formulas

$$B_{l+m}(x \bullet y) = Z(x,y) + (-1)^{|x||y|} Z(y,x), \tag{4.1}$$

$$\begin{aligned} (-1)^{|x|}\big(Z(x,y) - B_m(x) \bullet y\big) - x \bar{\circ} y \\ = (-1)^b \partial H(x,y) + H(\partial x, y) + (-1)^{l+b+1} H(x, \partial y), \end{aligned} \tag{4.2}$$

$$z \bullet w - (-1)^{|z||w|} w \bullet z = (-1)^{|z|}\big(\partial(z \bar{\circ} w) - (\partial z)\bar{\circ} w - (-1)^{|w|-1} z \bar{\circ} (\partial w)\big) \tag{4.3}$$

imply the equation

$$\begin{aligned} &B_{l+m}(x \bullet y) - \big(B_l(x) \bullet y + (-1)^{|x|} x \bullet B_m(y) + (-1)^{|x|}[x,y]\big) \\ &\quad = (-1)^{|x|+b}\big(\partial H(x,y) + (-1)^b H(\partial x, y) + (-1)^{l+1} H(x, \partial y)\big) \\ &\quad\quad + (-1)^{|x||y|+a}\big(\partial H(y,x) + (-1)^a H(\partial y, x) + (-1)^{m+1} H(y, \partial x)\big) \\ &\quad\quad - (-1)^{(|x|+1)|y|}\big(\partial(B_m(y)\bar{\circ} x) - (\partial B_m(y))\bar{\circ} x - (-1)^{|y|} B_m(y) \bar{\circ} (\partial x)\big). \end{aligned}$$

The formula (4.1) follows directly from the definition, and (4.3) is [15, (3.7)]. (4.2) follows from the following formulas, which are proved similarly as in [10, §6]:

$$H(d^0(x) + (-1)^{l+1} d^{l+1}(x), y) = (-1)^{|x|} Z(x,y) - x \bar{\circ} y,$$

$$\sum_{1\le j<p\le l} H_{j,p}((-1)^{p-j} d^{p-j}(x), y) = (-1)^{l+b} H(x, d^0 y),$$

$$\sum_{1\le j\le p\le l-1} H_{j,p}((-1)^{p-j+1} d^{p-j+1}(x), y) = (-1)^{l+b} H(x, (-1)^{m+1} d^{m+1}(y)),$$

$$\sum_{1\le j\le l} H_{j,l}((-1)^{l-j+1} d^{l-j+1}(x), y) = (-1)^{|x|+1} B(x) \bullet y,$$

$$\begin{aligned} &\sum_{1\le j\le p\le l} \sum_{\substack{1\le i\le l \\ i\ne p-j,\, p-j+1}} H_{j,p}((-1)^i d^i(x), y) \\ &\quad = (-1)^{b+1}\Big(\sum_{1\le j\le p\le l-1} \sum_{\substack{1\le i\le p-1, \text{ or} \\ p+m\le i\le l+m-2}} (-1)^i d^i(H(x,y)) \\ &\quad\quad + d^0(H(x,y)) + (-1)^{l+m-1} d^{l+m-1}(H(x,y))\Big), \end{aligned}$$

$$\sum_{1\le j\le p\le l-1} \sum_{p\le i\le p+m-1} (-1)^i d^i(H_{j,p}(x,y)) = (-1)^l \sum_{1\le i\le m} (-1)^i H(x, d^i(y)). \qquad \square$$

Corollary 4.7. *B_k defines a BV-algebra structure on $GrH_*(\mathrm{EC}(1, B^{n-1}))$, where Gr stands for the graded quotient.*

Proof. The cyclic structure on $f\mathcal{C}$ is described in [3]. An easy observation shows that $\tau_*\mu = \mu$ for the operad $H_*(f\mathcal{O})$, where the multiplication $\mu \in H_0(f(\mathcal{C}(2))) \cong \mathbb{Z}$ corresponds to $1 \in \mathbb{Z}$. Thus $H_*(f\mathcal{C})$ is a cyclic multiplicative operad and $E^2_{**} = HH_*(H_*(f\mathcal{O}))$ admits a BV-algebra structure.

The Bousfield spectral sequence [1] is derived from the double complex $C_*(f\mathcal{C}(*))$ with boundary operators d and ∂_k, where C_* is the singular chain complex functor and d is the boundary operation of C_*. Since B_k is defined on $C_*(f\mathcal{C}(*))$ and commutes with both d and ∂, B_* descends to a map on E^r, $r \geq 2$. □

Conjecture. *At least over rationals, B coincides with λ discussed in §3.*

Acknowledgment

The author is deeply grateful to Ryan Budney and Paolo Salvatore for informing him about their framed little disks action. The author is partially supported by Grant-in-Aid for JSPS Fellows, 228006.

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